11.3.2 Moment-Generating Function

Moment generatin

This last expression is the product of the moment generating-functions for M scaled Poisson processes, the mth process having intensity $\lambda p(y_m)$ and jump size y_m.

- If Y_i takes one of finitely many possible nonzero values $y_1, y_2, \ldots, y_m$, with $p(y_m) = P(Y_i = y_m)$ so that $p(y_m) > 0$ for every m and $\varphi_Y(u) = \sum_{m=1}^{M} P(y_m) e^{uy_m}$. It follows from (11.3.2)

$$Q(t) = \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0.$$

$$\begin{aligned}\varphi_{Q(t)}(u) &= \exp\left\{\lambda t\left(\sum_{m=1} p(y_m)e^{uy_m} - 1\right)\right\} \\ &= \exp\left\{\lambda t \sum_{m=1}^{M} p(y_m)\left(e^{uy_m} - 1\right)\right\} \\ &= \exp\{\lambda p(y_1)t(e^{uy_1} - 1)\}\exp\{\lambda p(y_2)t(e^{uy_2} - 1)\}\cdots \\ &\quad \cdots\exp\{\lambda p(y_M)t(e^{uy_M} - 1)\}. \end{aligned} \quad (11.3.5)$$

M scaled Poisson process:
跳y_m距離的$Poisson\ process$
有intensity $\lambda p(y_m)$

Moment generating function

Theorem 11.3.3 (Decomposition of a compound Poisson process). *Let $y_1, y_2, \ldots, y_M$ be a finite set of nonzero numbers, and let $p(y_1), p(y_2), \ldots, p(y_M)$ be positive numbers that sum to 1. Let $\lambda > 0$ be given, and let $\overline{N}_1(t), \overline{N}_2(t), \ldots, \overline{N}_M(t)$ be independent Poisson processes, each $\overline{N}_m(t)$ having intensity $\lambda p(y_m)$. Define*

$$\overline{Q}(t) = \sum_{m=1}^{M} y_m \overline{N}_m(t), \quad t \geq 0. \tag{11.3.6}$$

$\overline{N}_m(t)$: 第m種jump size跳$\overline{N}_m(t)$次

$\overline{Q}(t)$: jump size乘次數的總合

$\overline{Q}$(t) is a compound Poisson process.

Moment generating function

- If $\bar{Y}_1$ is the size of the first jump of $\bar{Q}$ (t), $\bar{Y}_2$ is the size of the second jump, etc. , and $\overline{N}(t) = \sum_{m=1}^{M} \overline{N}_m(t), \quad t \geq 0,$ is the total number of jumps on the interval (0, t].

- ⟹ $\bar{N}$ (t) is a Poisson process with intensity λ, the random variables $\bar{Y}_1, \bar{Y}_2, \ldots$ are independent with P{$\bar{Y}_i = y_m$} = $p(y_m)$ for m = 1 , ... ,M, the random variables $\bar{Y}_1, \bar{Y}_2, \ldots \bar{Y}_M$ are independent of $\bar{N}$ (t) , and

$$\overline{Q}(t) = \sum_{i=0}^{\overline{N}(t)} \overline{Y}_i, \quad t \geq 0.$$

Moment generating function

OUTLINE OF PROOF: According to (11.3.3), for each m, the characteristic function of $y_m \overline{N}_m(t)$ is

$$\varphi_{yN(t)}(u) = \mathbb{E}e^{uyN(t)} = \exp\{\lambda t(e^{uy} - 1)\}. \quad (11.3.3)$$

$$\varphi_{y_m \overline{N}_m(t)}(u) = \exp\{\lambda p(y_m)t(e^{uy_m} - 1)\}.$$

With $\overline{Q}(t)$ defined by (11.3.6), we use the fact that $\overline{N}_1(t), \overline{N}_2(t), \ldots, \overline{N}_M(t)$ are independent of one another to write

$$\overline{Q}(t) = \sum_{m=1}^{M} y_m \overline{N}_m(t), \quad t \geq 0. \quad (11.3.6)$$

$$\begin{aligned}
\varphi_{\overline{Q}(t)}(u) &= \mathbb{E}\exp\left\{u \sum_{m=1}^{M} y_m \overline{N}_m(t)\right\} \\
&= \mathbb{E}e^{uy_1\overline{N}_1(t)}\mathbb{E}e^{uy_2\overline{N}_2(t)} \cdots \mathbb{E}e^{uy_M\overline{N}_M(t)} \\
&= \varphi_{y_1\overline{N}_1(t)}(u)\varphi_{y_2\overline{N}_2(t)}(u) \cdots \varphi_{y_M\overline{N}_M(t)}(u) \\
&= \exp\{\lambda p(y_1)t(e^{uy_1} - 1)\}\exp\{\lambda p(y_2)t(e^{uy_2} - 1)\} \cdots \\
&\qquad \cdots \exp\{\lambda p(y_M)t(e^{uy_M} - 1)\},
\end{aligned}$$

$\overline{N}_1(t), \overline{N}_2(t), \ldots, \overline{N}_M(t)$ are independent

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Moment generating function

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- The random variable $\bar{Q}$(t) of (11.3.6) has the **same distribution** as the random variable Q(t) appearing on the left-hand side of (11.3.5) .

$$\varphi_{Q(t)}(u) = \exp\left\{\lambda t\left(\sum_{m=1}^{M} p(y_m)e^{uy_m} - 1\right)\right\}$$
$$= \exp\left\{\lambda t \sum_{m=1}^{M} p(y_m)(e^{uy_m} - 1)\right\}$$
$$= \exp\{\lambda p(y_1)t(e^{uy_1} - 1)\}\exp\{\lambda p(y_2)t(e^{uy_2} - 1)\}\cdots$$
$$\cdots\exp\{\lambda p(y_M)t(e^{uy_M} - 1)\}. \qquad (11.3.5)$$

$$\varphi_{\overline{Q}(t)}(u) = \mathbb{E}\exp\left\{u\sum_{m=1}^{M} y_m\overline{N}_m(t)\right\}$$
$$= \mathbb{E}e^{uy_1\overline{N}_1(t)}\mathbb{E}e^{uy_2\overline{N}_2(t)}\ldots\mathbb{E}e^{uy_M\overline{N}_M(t)}$$
$$= \varphi_{y_1\overline{N}_1(t)}(u)\varphi_{y_2\overline{N}_2(t)}(u)\cdots\varphi_{y_M\overline{N}_M(t)}(u)$$
$$= \exp\{\lambda p(y_1)t(e^{uy_1} - 1)\}\exp\{\lambda p(y_2)t(e^{uy_2} - 1)\}\cdots$$
$$\cdots\exp\{\lambda p(y_M)t(e^{uy_M} - 1)\},$$

Moment generating function

- The substance of Theorem 11.3.3 is that there are two equivalent ways of regarding a compound Poisson process that has only finitely many possible jump sizes.

- First, it can be thought of as a single Poisson process in which the size-one jumps are replaced by jumps of random size.

$$Q(t) = \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0. \qquad (11.3.1)$$

- Alternatively, it can be regarded as a sum of independent Poisson processes in each of which the size-one jumps are replaced by jumps of a fixed size.

$$\overline{Q}(t) = \sum_{m=1}^{M} y_m \overline{N}_m(t), \quad t \geq 0. \qquad (11.3.6)$$

Moment generating function

Corollary 11.3.4. *Let $y_1, \ldots, y_M$ be a finite set of nonzero numbers, and let $p(y_1), \ldots, p(y_M)$ be positive numbers that sum to 1. Let $Y_1, Y_2, \ldots$ be a sequence of independent, identically distributed random variables with $\mathbb{P}\{Y_i = y_m\} = p(y_m)$, $m = 1, \ldots, M$. Let $N(t)$ be a Poisson process and define the compound Poisson process*

$$Q(t) = \sum_{i=1}^{N(t)} Y_i.$$

For $m = 1, \ldots, M$, let $N_m(t)$ denote the number of jumps in Q of size y_m up to and including time t. Then

$$N(t) = \sum_{m=1}^{M} N_m(t) \text{ and } Q(t) = \sum_{m=1}^{M} y_m N_m(t).$$

The processes $N_1, \ldots, N_M$ defined this way are independent Poisson processes, and each N_m has intensity $\lambda p(y_m)$.